Chapter 3. Renewal Processes
Outline

• Distribution and Limiting Behavior of $N(t)$
  – Pmf of $N(t)$: $P(N(t) = k) = ?$
  – Limiting time average: $\lim_{t \to \infty} \frac{N(t)}{t} = ?$ (Law of Large Numbers)
  – Limiting PDF of $N(t)$ (Central Limit Theorem)

• Renewal Function $E[N(t)]$, and its Asymptotic (Limiting) behavior
  – Renewal Equation
  – Wald’s Theorem and Stopping time
  – Elementary Renewal Theorem
  – Blackwell’s Theorem
Outline

• Key Renewal Theorem and Applications
  – Definition of Regenerative Process
  – Renewal Theory
  – Application: Residual Life, Age, and Total Life

• Renewal Reward Processes and Applications
  – Renewal Reward Process/Theory
  – Application 1: Alternating Renewal Process/Theory
  – Application 2: Time Average of Residual Life and Age
Distribution and Limiting Behavior of $N(t)$

- $\{X_n, n = 1, 2, \ldots\}$ are iid RVs with distribution $F(t)$: mean $\mu$ ($0 < \mu < \infty$)

- $N(t)$ represents the number of renewals in $(0, t]$. The stochastic process $\{N(t), t \geq 0\}$ is called a renewal (counting) process

$$N(t) = \sup\{n : S_n \leq t\} (\therefore \text{There are always finite renewals})$$

$$= \max\{n : S_n \leq t\} \text{ in a finite time (i.e., } N(t) < \infty)$$
Distribution and Limiting Behavior of $N(t)$

1. pmf of $N(t) \rightarrow$ closed-form

2. Limiting time average [Law of Large Numbers]:

$$\frac{N(t)}{t} \xrightarrow{w.p.1} \frac{1}{\mu}, t \rightarrow \infty$$

3. Limiting PDF of $N(t)$ [Central Limit Theorem]:

$$P\left\{ \frac{N(t) - t/\mu}{\sigma \sqrt{t/\mu^3}} < y \right\}$$

$$\rightarrow \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \sim Gaussian\left(\frac{t}{\mu}, \sigma \sqrt{t} \cdot \mu^{-\frac{3}{2}}\right) \quad \text{as } t \rightarrow \infty$$
4. Limiting time and ensemble average
   [Elementary Renewal Theorem]:

   \[ \frac{E[N(t)]}{t} \xrightarrow{w.p.1} \frac{1}{\mu}, t \to \infty \]

5. Limiting ensemble average (focusing on arrivals in the vicinity of \( t \))
   [Blackwells Theorem]

   \[ \frac{E[N(t + \delta) - N(t)]}{\delta} \xrightarrow{w.p.1} \frac{1}{\mu}, t \to \infty \]
1. pmf of $N(t)$

\[ \int_0^t \varphi(s)\psi(t-s) \, ds, \quad 0 \leq t < \infty, \]

\[ \therefore \quad \{N(t) \geq n\} \iff \{S_n \leq t\} \text{ (fundamental relationship)} \]

\[ \therefore \quad P\{N(t) \geq n\} = P\{S_n \leq t\} = F_n(t) \quad \text{and} \]

\[ P\{N(t) = n\} = P\{N(t) \geq n\} - P\{N(t) \geq n + 1\} = P\{S_n \leq t\} - P\{S_{n+1} \leq t\} = F_n(t) - F_{n+1}(t) \]

\[ F_n(t) = P\{S_n \leq t\} = P\left\{ \sum_{i=1}^{n} X_i \leq t \right\} \]

\[ \therefore \quad X_i \sim F \]

\[ \therefore \quad \sum X_i \sim F(t) \otimes \ldots \otimes F(t) \equiv F_n(t) \text{ (n-fold convolution)} \]
Example

Density of $N(t)$ for Poisson Process

Let $N$ be a Poisson process with rate $\lambda$. Recall that the sum of i.i.d. Exponential random variables is an Erlang random variable, such that

$$F_n(t) = \int_0^t \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \, dt$$

What is the pmf of $N(t)$?

$$P\{N(t) = n\} = F_n(t) - F_{n+1}(t)$$

$$= \int_0^t \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \, dt - \int_0^t \frac{\lambda^{n+1} t^n e^{-\lambda t}}{n!} \, dt$$

$$= \sum_{k=0}^n \frac{(\lambda t)^k e^{-\lambda t}}{k!} - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$= \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$
2. Limiting Time Average

\[ \lim_{t \to \infty} N(t) = ? \]

\[ \therefore P\{ \lim_{t \to \infty} N(t) < \infty \} = P\{ N(\infty) < \infty \} = P\{ X_n = \infty \text{ for some } n \} \]

\[ = P\left[ \bigcup_{n=1}^{\infty} \{ X_n = \infty \} \right] \leq \sum_{n=1}^{\infty} P\{ X_n = \infty \} = 0 \]

\[ \therefore \lim_{t \to \infty} N(t) = N(\infty) = \infty \quad w.p.1 \]

Whether an infinite number of renewals can occur in a finite time?

\[ \frac{S_n}{n} \to \mu \quad \text{as } n \to \infty \quad \text{strong law of large number} \]

\[ \therefore \mu > 0 \quad \text{As } n \to \infty, S_n \to \infty \]

\[ \therefore N(t) = \max\{ n : S_n \leq t \} \]
Strong Law of Large Numbers

If \( X_1, X_2, \ldots \) is an i.i.d. random sequence with sample mean \( M_n(X) \),

\[
\lim_{n \to \infty} M_n(X) = E[X] \text{ w.p. 1}
\]

w.p. = with probability

\[
P\left(\lim_{n \to \infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = \mu\right) = 1
\]
2. Limiting Time Average (con’t)

**Question:** What is the rate at which $N(t)$ goes to $\infty$? 

\[ \text{i.e., } \lim_{t \to \infty} \frac{N(t)}{t} =? \]
**Strong Law for Renewal Processes**

**Theorem.** For a renewal process \( \{N(t), t \geq 0\} \) with mean inter-renewal interval \( \mu \), then

\[
\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu}, \quad \text{w.p.} 1
\]

**Proof.**

\[
\therefore \quad S_{N(t)} \leq t < S_{N(t)+1} \\
\therefore \quad \frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}
\]

Average of the first \( N(t) \) inter-arrival times

\[
\frac{S_{N(t)}}{N(t)} \to \mu \quad \text{as} \quad N(t) \to \infty
\]

\[
\Rightarrow \frac{S_{N(t)}}{N(t)} \to \mu \quad \text{as} \quad t \to \infty
\]

\[
\lim_{n \to \infty} M_n(X) = E[X] \quad \text{w.p.} 1
\]

**strong law of large number**

\[
N(t) \to \infty, \quad S_{N(t)} \to \infty \Rightarrow t \to \infty
\]
\[
\frac{S_{N(t)+1}}{N(t)} = \left[ \frac{S_{N(t)+1}}{N(t) + 1} \right] \left[ \frac{N(t) + 1}{N(t)} \right]
\]

\Rightarrow \frac{S_{N(t)+1}}{N(t)} \to \mu \quad \text{as } t \to \infty

\Rightarrow \frac{t}{N(t)} \to \mu \quad \text{as } t \to \infty

\Rightarrow \frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{as } t \to \infty

\#
Central Limit Theorem

Given $X_1, X_2, \ldots$, a sequence of iid random variables with expected value $\mu_X$ and variance $\sigma_X^2$, the CDF of $Z_n = (\sum_{i=1}^{n} X_i - n\mu_X)/\sqrt{n}\sigma_X^2$ has the property

$$\lim_{n \to \infty} F_{Z_n}(z) = \Phi(z).$$

The PMF of the $X$, the number of heads in $n$ coin flips for $n = 5, 10, 20$. As $n$ increases, the PMF more closely resembles a bell-shaped curve.
3. Central Limit Theorem for $N(t)$

**Theorem.** Assume that the inter-renewal intervals for a renewal process $\{N(t), t \geq 0\}$ have finite mean and variance $\mu, \sigma^2$. Then,

$$
\lim_{t \to \infty} P \left[ \frac{N(t) - t/\mu}{\sigma \sqrt{t/\mu^3}} < y \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} \, dx
$$

**Proof.** (idea: $N(t) \to S_{N(t)} \to CLT$)

Let $r_t = t/\mu + y\sigma \sqrt{t/\mu^3}$. Then

$$
P\left\{ \frac{N(t) - t/\mu}{\sigma \sqrt{t/\mu^3}} < y \right\} = P\{N(t) < r_t\} = P\{S_{r_t} > t\}
$$

$$
= P\left\{ \frac{S_{r_t} - r_t \mu}{\sigma \sqrt{r_t}} > \frac{t - r_t \mu}{\sigma \sqrt{r_t}} \right\}
$$
\[ P\left\{ \frac{S_{r_t} - r_t \mu}{\sigma \sqrt{r_t}} > -y \sqrt{\frac{\sqrt{t \mu}}{y \sigma + \sqrt{t \mu}}} \right\} \to 1 \quad \text{as } t \to \infty \]

\[ = P\left\{ \frac{S_{r_t} - r_t \mu}{\sigma \sqrt{r_t}} > -y \right\} = 1 - \Phi(-y) = \Phi(y) \]

\[ \mathcal{N}(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

\[ \Rightarrow P\left\{ \frac{N(t) - t/\mu}{\sigma \sqrt{t/\mu^3}} < y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \, dx \]

\[ \sim Gaussian\left(\frac{t}{\mu}, \sigma \sqrt{t} \cdot \mu^{-\frac{3}{2}}\right) \quad \text{as } t \to \infty \]
4. Renewal Function $E[N(t)]$

Let $m(t) = E[N(t)]$, which is called "renewal function".

1. Relationship between $m(t)$ and $F_n$

   $$m(t) = \sum_{n=1}^{\infty} F_n(t), \text{ where } F_n(t) \text{ is the } n\text{-fold convolution of } F(t)$$

2. Relationship between $m(t)$ and $F(t)$
   
   [Renewal Equation]

   $$m(t) = F_X(t) + \int_0^t m(t-x) \cdot dF_X(x)$$
3. Relationship between $m(t)$ and $\tilde{F}(s)$ ($\tilde{F}(s)$: Laplace Transform of $F(t)$)

$$\tilde{m}(s) = \frac{\tilde{F}(s)}{1 - \tilde{F}(s)}$$

→ [Wald’s Equation]

4. Asymptotic behavior of $m(t)$ ($t \to \infty$, Limiting)
→ [Elementary Renewal Theorem]
→ [Blackwell’s Theorem]
4. Renewal Function $E[N(t)]$ (con’t)

1. $m(t) = E[N(t)] \leftrightarrow F_n(t)$ (i.e., CDF of $S_n$)

Reminding that

$$P\{N(t) \geq n\} = P\{S_n \leq t\} = F_n(t)$$

and

$$P\{N(t) = n\} = P\{N(t) \geq n\} - P\{N(t) \geq n + 1\}$$
$$\quad = P\{S_n \leq t\} - P\{S_{n+1} \leq t\}$$
$$\quad = F_n(t) - F_{n+1}(t)$$

$$\sum_{n=1}^{\infty} F_n(t) = \sum_{n=1}^{\infty} P\{N(t) \geq n\} = P\{N(t) \geq 1\} + P\{N(t) \geq 2\} + \ldots$$

$$P\{N(t) = 1\} + P\{N(t) = 2\} + \ldots$$

$$P\{N(t) = 2\} + P\{N(t) = 3\} + \ldots$$
\[
\sum_{n=1}^{\infty} F_n(t) = \sum_{n=1}^{\infty} P\{N(t) \geq n\} = P\{N(t) \geq 1\} + P\{N(t) \geq 2\} + \ldots
\]

\[
= [P\{N(t) = 1\} + P\{N(t) = 2\} + P\{N(t) = 3\} + \ldots] \\
+ [P\{N(t) = 2\} + P\{N(t) = 3\} + P\{N(t) = 4\} + \ldots] \\
+ [P\{N(t) = 3\} + P\{N(t) = 4\} + P\{N(t) = 5\} + \ldots] \\
+ \ldots
\]

\[
= 1P\{N(t) = 1\} + 2P\{N(t) = 2\} + 3P\{N(t) = 3\} + \ldots
\]

\[
= \sum_{n=1}^{\infty} nP\{N(t) = n\}
\]

\[
= E[N(t)]
\]

\[
\Rightarrow m(t) = E[N(t)] = \sum_{n=1}^{\infty} F_n(t)
\]
4. Renewal Function $E[N(t)]$ (con’t)

1. $m(t) = E[N(t)] \leftrightarrow F_n(t)$ (i.e., CDF of $S_n$)

Let $N(t) = \sum_{n=1}^{\infty} I_n$, where $I_n = \begin{cases} 1, & \text{if the } n_{th} \text{ renewal occurred in } [0,t] \\ 0, & \text{Otherwise;} \end{cases}$

$$E[N(t)] = E[\sum_{n=1}^{\infty} I_n]$$

$$= \sum_{n=1}^{\infty} E[I_n]$$

$$= \sum_{n=1}^{\infty} P\{I_n = 1\}$$

$$= \sum_{n=1}^{\infty} P\{S_n \leq t\}$$

$$= \sum_{n=1}^{\infty} F_n(t)$$

Proof method in p.100 of the textbook
4. Renewal Function $E[N(t)]$ (con’t)

$$m(t) = \sum_{n=1}^{\infty} F_n(t)$$

$F_n(t)$: CDF of $S_n$

$n$-fold convolution $F(t)$

As $t \to \infty$, $n \to \infty$, finding $F_n$ is far too complicated
\Rightarrow find another way of solving $m(t)$ in terms of $F(t)$

Big Problem!!!
4. Renewal Function $E[N(t)]$ (con’t)

2. $m(t) \leftrightarrow F(t)$ (i.e., CDF of $X_i$)

\[
\therefore S_n = S_{n-1} + X_n, \text{ for all } n \geq 1, \text{ and } S_{n-1} \text{ and } X_n \text{ are independent,}
\]

\[
\therefore \text{For } n \geq 2, \ P[S_n \leq t] = \int_0^t P[S_{n-1} \leq t - x]dF_X(x)
\]

For $n = 1$, $X_1 = S_1$, $P[S_1 \leq t] = F_X(t)$

\[
\therefore m(t) = \sum_{n=1}^{\infty} P[S_n \leq t] = F_X(t) + \sum_{n=2}^{\infty} \int_0^t P[S_{n-1} \leq t - x]dF_X(x)
\]

\[
\Rightarrow \ m(t) = F_X(t) + \int_0^t m(t - x)dF_X(x) \quad \text{Renewal Equation}
\]
4. Renewal Function $E[N(t)]$ (con’t)

3. $\tilde{m}(s) \leftrightarrow \tilde{F}(s)$
   $\tilde{F}(s)$ : Laplace Transform of $F$
   $\tilde{m}(s)$ : Laplace Transform of $m(t)$

   Answer:

   Proof:
   
   $$m(t) = \sum_{n=1}^{\infty} F_n(t) \Rightarrow \tilde{m}(s) = \sum_{n=1}^{\infty} \tilde{F}_n(s)$$
   
   $$= \sum_{n=1}^{\infty} (\tilde{F}(s))^n$$
   
   $$= \frac{\tilde{F}(s)}{1 - \tilde{F}(s)}$$

   Why?
   See next slide.
Laplace Transform of $f * g$

the convolution of signals $f$ and $g$, denoted $h = f * g$, is the signal

$$h(t) = \int_0^t f(\tau)g(t - \tau) \, d\tau$$

- same as $h(t) = \int_0^t f(t - \tau)g(\tau) \, d\tau$; in other words,

  $$f * g = g * f$$

- (very great) importance will soon become clear

in terms of Laplace transforms:

$$H(s) = F(s)G(s)$$

Laplace transform turns convolution into multiplication
Laplace Transform of $f \ast g$ (con’t)

let's show that $\mathcal{L}(f \ast g) = F(s)G(s)$:

\[
H(s) = \int_{t=0}^{\infty} e^{-st} \left( \int_{\tau=0}^{t} f(\tau)g(t-\tau) \, d\tau \right) \, dt
\]

\[
= \int_{t=0}^{\infty} \int_{\tau=0}^{t} e^{-st}f(\tau)g(t-\tau) \, d\tau \, dt
\]

where we integrate over the triangle $0 \leq \tau \leq t$

- change order of integration: $H(s) = \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} e^{-st}f(\tau)g(t-\tau) \, dt \, d\tau$

- change variable $t$ to $\bar{t} = t - \tau; \, d\bar{t} = dt$; region of integration becomes $\tau \geq 0, \, \bar{t} \geq 0$

\[
H(s) = \int_{\tau=0}^{\infty} \int_{\bar{t}=0}^{\infty} e^{-s(\bar{t}+\tau)}f(\tau)g(\bar{t}) \, d\bar{t} \, d\tau
\]

\[
= \left( \int_{\tau=0}^{\infty} e^{-s\tau}f(\tau) \, d\tau \right) \left( \int_{\bar{t}=0}^{\infty} e^{-s\bar{t}}g(\bar{t}) \, d\bar{t} \right)
\]

\[
= F(s)G(s)
\]
4. Renewal Function $E[N(t)]$ (con’t)

4. Asymptotic behavior of $m(t)$:

$$\lim_{t \to \infty} \frac{m(t)}{t} = \lim_{t \to \infty} \frac{E[N(t)]}{t} = \frac{1}{\mu}$$

Elementary Renewal Theorem

Before proving elementary renewal theorem, we first introduce the stopping time (or stopping rule) and Wald’s equation.
Stopping Time (Rule)

**Definition.** $N$, an integer-valued r.v., is said to be a ”stopping time” for a set of independent r.v. $X_1, X_2, \ldots$ if event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$

**Example.**

- Let $X_1, X_2, \ldots$ be independent random variables
- $P[X_n = 0] = P[X_n = 1] = 1/2, \quad n = 1, 2, \ldots$
- If $N = \min\{n : X_1 + \ldots + X_n = 10\}$
  $\rightarrow$ Is $N$ a stopping time for $X_1, X_2, \ldots$? **Answer:** Yes

**Example.**

Let $X_n, n=1, 2, \ldots$ be independent and let $N = \max\{n : X_n \geq 5\}$

Then, is $N$ a stopping time? **Answer:** No
Stopping Time (Rule)

Example.

- $N(t), \{X_n, n = 1, 2, 3, \ldots\}$,
- $\{S_n, n = 0, 1, 2, 3, \ldots\}$
- $S_n = S_{n-1} + X_n$

→ Is $N(t)$ the stopping time of $\{X_n, n = 1, 2, \ldots\}$?

Answer:
For a renewal process $\{N(t), t \geq 0\}$, $N(t)$ is not a stopping time for interarrival sequence $X_i$'s.
Why? [Homework]
Example. Is $N(t) + 1$ the stopping time for $\{X_n\}$?

Answer:

For a renewal process, $N(t) + 1$ is a stopping time for the interarrival sequence $X_1, X_2, \ldots$ It can be seen that the following events are equivalent:

$$\{N(t) + 1 = n\} \equiv \{N(t) = n - 1\} \equiv \{X_1 + \ldots + X_{n-1} \leq t \text{ and } X_1 + \ldots + X_n > t\}$$

So, $\{N(t) + 1 = n\}$ depends only on $X_1, \ldots, X_n$ and is independent of $X_{n+1}, X_{n+2}, \ldots$
Examples. To illustrate some examples of random times that are stopping rules and some that are not, consider a gambler playing roulette with a typical house edge, starting with $100:

- Playing one, and only one, game corresponds to the stopping time $\tau = 1$, and is a stopping rule.
- Playing until he either runs out of money or has played 500 games is a stopping rule.
- Playing until he is the maximum amount ahead he will ever be is not a stopping rule and does not provide a stopping time, as it requires information about the future as well as the present and past.
- Playing until he doubles his money (borrowing if necessary if he goes into debt) is not a stopping rule, as there is a positive probability that he will never double his money. (Here it is assumed that there are limits that prevent the employment of a martingale system or variant thereof (such as each bet being triple the size of the last). Such limits could include betting limits but not limits to borrowing.)
- Playing until he either doubles his money or runs out of money is a stopping rule, even though there is potentially no limit to the number of games he plays, since the probability that he stops in a finite time is 1
Wald's Equation

**Theorem.** If $X_1, X_2, \ldots$ are i.i.d. random variables with finite mean $E[X]$, and if $N$ is the stopping time for $X_1, X_2, \ldots$, such that $E[N] < \infty$. Then,

$$E \left[ \sum_{n=1}^{N} X_n \right] = E[N] E[X]$$

**Proof.**

Let $I_n = \begin{cases} 1, & \text{if } N \geq n; \\ 0, & \text{Otherwise; } \end{cases}$

we have

$$\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} X_n I_n$$

$$\Rightarrow \quad E[\sum_{n=1}^{N} X_n] = E[\sum_{n=1}^{\infty} X_n I_n] = \sum_{n=1}^{\infty} E[X_n I_n]$$
\[ \therefore X_n \text{ and } I_n \text{ are independent} \]

\[
E\left[ \sum_{n=1}^{N} X_n \right] = \sum_{n=1}^{\infty} E[X_n] E[I_n] \\
= E[X] \sum_{n=1}^{\infty} E[I_n] \\
= E[X] \sum_{n=1}^{\infty} P\{N \geq n\} \\
= E[X] E[N]
\]

\[
\sum_{n=1}^{\infty} P\{N \geq n\} = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P\{N = k\} \\
= \sum_{k=1}^{\infty} \sum_{n=1}^{k} P\{N = k\} \\
= \sum_{k=1}^{\infty} kP\{N = k\} \\
= E[N]
\]

#

For Wald’s Theorem to be applied, other than \(X_1, X_2, \ldots\):

1. \(N\) must be a stopping time; and
2. \(E[N] < \infty\)
Example (Simple Random Walk)

\{X_i\} i.i.d. with: \[P(X = 1) = p\]
\[P(X = -1) = 1 - p = q\]

\[S_n = \sum_{k=1}^{n} X_k\]

Let \(N = \min\{n|S_n = 1\}\). Thus \(N\) is the first time the random sum reaches 1.

Find \(E[N]\)
Solution

$N$ is a stopping time with respect to $\{X_i\}$ since the event $\{N = n\}$ is determined solely by the sequence $\{X_1, ..., X_n\}$.

Since $E[|X_1|] < \infty$,

$$E[S_N] = (p - q)E[N].$$

(Wald’s equation)

$S_N = 1$ for all $N$ and $E[S_N] = 1$.

case $p = q$ and $E[N] < \infty$: right side is zero (does not make sense)

case $q < p$: right side negative and left side is 1.

$\Rightarrow$ case $p > q$: $E[N] = 1/(p - q)$ and $P\{N < \infty\} = 1$
4. Asymptotic behavior of \( m(t) \):

\[
\lim_{t \to \infty} \frac{m(t)}{t} = \lim_{t \to \infty} \frac{E[N(t)]}{t} = \frac{1}{\mu}
\]

**Elementary Renewal Theorem**

- Distribution and Limiting Behavior of \( N(t) \)
  - Pmf of \( N(t) \): \( P(N(t) = k) = ? \)
  - Limiting time average: \( \lim_{t \to \infty} \frac{N(t)}{t} = ? \) (Law of Large Numbers)
  - Limiting PDF of \( N(t) \) (Central Limit Theorem)

**Theorem.** For a renewal process \( \{N(t), t \geq 0\} \) with mean inter-renewal interval \( \mu \), then

\[
\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu}, \quad w.p.1
\]
Example 7.4  Let $U$ be a random variable which is uniformly distributed on $(0, 1)$; and define the random variables $Y_n, n \geq 1$, by

$$Y_n = \begin{cases} 
0, & \text{if } U > 1/n \\
n, & \text{if } U \leq 1/n
\end{cases}$$

$$Y_n \to 0 \quad \text{as } n \to \infty$$

$$E[Y_n] = n P\left\{ U \leq \frac{1}{n} \right\} = n \frac{1}{n} = 1$$

Therefore, even though the sequence of random variables $Y_n$ converges to 0, the expected values of the $Y_n$ are all identically 1. ■
4. Asymptotic behavior of $m(t)$:

$$\lim_{t \to \infty} \frac{m(t)}{t} = \lim_{t \to \infty} \frac{E[N(t)]}{t} = \frac{1}{\mu}$$

Elementary Renewal Theorem

A simple consequence? Not always!!!

**Theorem.** For a renewal process $\{N(t), t \geq 0\}$ with mean inter-renewal interval $\mu$, then

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu}, \quad \text{w.p.1}$$
Corollary  [Ross Corollary 3.3.3 ]

Before proving \( \lim_{t \to \infty} \frac{m(t)}{t} \to \frac{1}{\mu}, \)

**Corollary.** If \( \mu < \infty \), then

\[
E[S_{N(t)+1}] = \mu [m(t) + 1]
\]

**Proof.**

Let’s stop at the first renewal after \( t \), i.e., at \( N(t) + 1 \) renewal. To verify \( N(t) + 1 \) is a stopping time for the sequence of \( X_i \),

\[
N(t) + 1 = n \iff N(t) = n - 1
\]

\[
\iff X_1 + ... + X_{n-1} \leq t, X_1 + ... + X_n > t
\]

\( \{N(t) + 1 = n\} \) depends only on \( X_1, ... X_n \) and is independent of \( X_{n+1} \)

\( \Rightarrow N(t) + 1 \) is a stopping time. From Wald’s equation, when \( E[X] < \infty \),

\[
E[X_1 + ...X_{N(t)+1}] = E[X] E[N(t) + 1] \iff E[S_{N(t)+1}] = \mu [m(t) + 1]
\]
Elementary Renewal Theorem

Proof (1).

\[ \frac{m(t)}{t} \to \frac{1}{\mu} \quad \text{as} \ t \to \infty \]

\[ S_{N(t)+1} > t \]

\[ \Rightarrow \quad E[S_{N(t)+1}] = \mu [m(t) + 1] > t \quad \text{(Corollary 3.3.3)} \]

\[ \Rightarrow \quad \frac{[m(t) + 1]}{t} > \frac{1}{\mu} \]

\[ \Rightarrow \quad \liminf_{t \to \infty} \frac{m(t)}{t} \geq \frac{1}{\mu} \quad \text{------------------------} \quad (1) \]

For a constant \( M \), we define a new renewal process \( \{\tilde{X}_n, n = 1, 2, \ldots\} \), where

\[ \tilde{X}_n = \begin{cases} X_n, & \text{if} \ X_n \leq M \\ M, & \text{if} \ X_n > M \end{cases} \]
Let \( \tilde{S}_n = \sum_{1}^{n} \tilde{X}_i \) and \( \tilde{N}_t = \sup\{n : \tilde{S}_n \leq t\} \)

**Corollary**

\[
E[S_{N(t)+1}] = \mu[m(t) + 1]
\]

\[
\tilde{S}_{N(t)+1} \leq t + M
\]

\[\Rightarrow (\tilde{m}(t) + 1) \mu M \leq t + M, \text{ where } \mu M = E[\tilde{X}_n]\]

\[\Rightarrow \limsup_{t \to \infty} \frac{\tilde{m}(t)}{t} \leq \frac{1}{\mu M}\]

\[\therefore \tilde{S}_n \leq S_n \quad \therefore \tilde{N}(t) \geq N(t) \text{ and } \tilde{m}(t) \geq m(t)\]

\[\Rightarrow \limsup_{t \to \infty} \frac{m(t)}{t} \leq \frac{1}{\mu M}\]

Let \( M \to \infty \Rightarrow \mu M = \mu \)

\[\Rightarrow \limsup_{t \to \infty} \frac{m(t)}{t} \leq \frac{1}{\mu}\]

\[\text{(2)}\]

From (1) and (2), we have

\[
\frac{m(t)}{t} \to \frac{1}{\mu} \text{ as } t \to \infty
\]
Elementary Renewal Theorem

Proof (2).

\[ t < \sum_{i=1}^{N(t)+1} X_i < t + M. \]

\[ t < \mu(m(t) + 1) < t + M, \]

or

\[ t - \mu < \mu m(t) < t + M - \mu, \]

or

\[ \frac{1}{\mu} - \frac{1}{t} < \frac{m(t)}{t} < \frac{1}{\mu} + \frac{M - \mu}{\mu t}. \]

Let \( t \to \infty \) to see that \( \frac{m(t)}{t} = \frac{1}{\mu} \).

\[ \frac{m(t)}{t} \to \frac{1}{\mu} \quad \text{as} \quad t \to \infty \]

(Corollary 3.3.3)

\[ E[S_{N(t)+1}] = \mu[m(t) + 1] \]
Example 7.5  Beverly has a radio that works on a single battery. As soon as the battery in use fails, Beverly immediately replaces it with a new battery. If the lifetime of a battery (in hours) is distributed uniformly over the interval (30, 60), then at what rate does Beverly have to change batteries?

Solution:

If we let \( N(t) \) denote the number of batteries that have failed by time \( t \),

\[
\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu} = \frac{1}{45}
\]
Example (M/G/1 Loss)

Customers arrive at a telephone booth according to a Poisson process having rate $\lambda$. However, a customer only enters the booth when it is empty. So, *no* queues are allowed. The service time (time duration that a customer occupies the booth) follows a general distribution $G$. This type of queueing system is called M/G/1 loss system.

1. What is the rate at which customers enter the booth?

2. What proportion of potential customers cannot enter the booth (called loss rate)?

3. What is the utilization of the booth?
Solution

A renewal occurs every time that a customer actually enters the booth.
If \( N(t) \) denotes the number of customers who enter the booth by \( t \), then \( \{N(t) \, , \, t \geq 0\} \) will be a renewal process having mean interarrival time of

\[
\mu = \text{mean service time} + \text{mean time to next customer's arrival}
\]

Since the time to the next customer follows exponential with rate \( \lambda \) from the memoryless property. So,

\[
\mu = \mu_G + \frac{1}{\lambda} \quad \text{where} \quad \mu_G = \int_{0}^{\infty} x dG(x).
\]

(1) Since the rate at which customers enter the booth is just the long-run renewal rate, it is given by

\[
\text{rate at which customers enter} = \frac{1}{\mu} = \frac{\lambda}{1 + \lambda \mu_G}
\]
Solution (con’t)

(2) The loss rate of customers is obtained by

\[
\text{loss rate} = 1 - \text{proportion of customers that enter} \\
\text{proportion of customers that enter} = \frac{\text{rate at which customers enter}}{\text{arrival rate}} \\
= \frac{\lambda}{(1 + \lambda \mu_G)} = \frac{1}{1 + \lambda \mu_G}
\]

So,

\[
\text{loss rate} = \frac{\lambda \mu_G}{1 + \lambda \mu_G} \quad \#
\]

(3) The utilization of the booth is

\[
\frac{\mu_G}{\mu_G + \frac{1}{\lambda}} \quad \#
\]
Lattice

- Definitions.

* A nonnegative random variable $X$ is said to be *lattice* if there exists $d \geq 0$ such that

$$\sum_{n=0}^{\infty} P[X = nd] = 1$$

* That is, $X$ is lattice if it only takes on integral multiples of some nonnegative number $d$. The largest $d$ having this property is said to be the *period* of $X$. If $X$ is lattice and $F$ is the distribution function of $X$, then we say that $F$ is *lattice*. 
Example. If each inter-renewal interval \( \{X_i, i = 1, 2, \ldots\} \) takes on integer number of time units, e.g., 0, 4, 8,12,..., then expected rate of renewals is zero at other times. Such random variable is said to be "lattice".

More examples.

(a) If \( X \) follows Poisson distribution with mean \( \lambda \), then \( X \) is lattice with period 1.
(b) If \( P\{X=4\}=P\{X=8\}=P\{X=12\}=P\{X=14\}=1/4 \), then \( X \) is lattice with period 2.
(c) If \( P\{X=2\pi\}=1/3 \) and \( P\{X=6\pi\}=2/3 \), then \( X \) is lattice with period \( 2\pi \).
(d) If \( P\{X=\sqrt{2}\}=P\{X=\sqrt{3}\}=1/2 \), then \( X \) is not lattice.
Background of Blackwell's Theorem

• Ensemble Average.
  – to determine the expected renewal rate in the limit of large $t$, without averaging from $0 \to t$ (time average)

• Question.
  – are there some values of $t$ at which renewals are more likely than others for large $t$ ?
Blackwell's Theorem

**Theorem.** (Proof in Feller)

- If, for \( \{X_i, i \geq 1\} \), which are not lattice, then, for any \( a \geq 0 \),

  \[
  \lim_{t \to \infty} m(t + a) - m(t) = \frac{a}{\mu} \quad \Rightarrow \quad \lim_{t \to \infty} \frac{d}{dt} \frac{m(t)}{dt} = \frac{1}{\mu}
  \]

- If the inter-renewal distribution is lattice with period \( d \), then for any integer \( n \geq 1 \),

  \[
  \lim_{n \to \infty} E[\text{number of renewals at time } nd] = \frac{d}{\mu} \quad \Rightarrow \quad \frac{\lim_{t \to \infty} E[N(t + kd) - N(t)]}{kd} = \frac{\lim_{t \to \infty} m(t + kd) - m(t)}{kd} = \frac{1}{\mu}, \quad k \geq 1
  \]

- Since the number of renewals at time \( nd \) will be 1 or 0,

  \[
  \lim_{n \to \infty} P\{\text{a renewal arrives at } nd\} \Rightarrow \frac{d}{\mu}
  \]
Example

Example. Suppose that interarrival times have the following distribution.

\[ X = \begin{cases} 
2 & \text{w.p. } 1/2 \\
4 & \text{w.p. } 1/2 
\end{cases} \]

1. Is \( X \) a lattice? What's its period?

2. What is the expectation of the number of renewals at \( 2n \)?

Solution

\( X \) is lattice with \( d=2 \). So,

\[
\lim_{n \to \infty} E[\text{number of renewals at } 2n] = \lim_{n \to \infty} P[\text{renewal occurs at } 2n] = \frac{2}{3}
\]
Directly Riemann Integrable

Let \( \overline{m}_n(a) \) and \( \underline{m}_n(a) \) be the sup and inf of \( f(t) \), i.e.,

\[
\overline{m}_n(a) = \sup\{f(t) : na \leq t < (n + 1)a\}
\]
\[
\underline{m}_n(a) = \inf\{f(t) : na \leq t < (n + 1)a\}
\]

Definition. \( f(t) \) is directly Riemann integrable if

\[
\sum_{n=0}^{\infty} \overline{m}_n(a) \text{ and } \sum_{n=0}^{\infty} \underline{m}_n(a) \text{ are finite, and}
\]
\[
\lim_{a \to 0} a \cdot \sum_{n=0}^{\infty} [\overline{m}_n(a) - \underline{m}_n(a)] = 0
\]
Directly Riemann Integrable (con’t)

• Sufficient conditions for an $h(t)$ to be directly Riemann integrable
  1. $h(t) \geq 0$ for all $t \geq 0$
  2. $h(t)$ is non-increasing
  3. $\int_{0}^{\infty} h(t) \, dt < \infty$
Key Renewal Theorem

Theorem.
If $F_X$ is non-lattice, and if $h(t)$ is directly Riemann integrable, then,

$$\lim_{t \to \infty} \int_0^t h(t - x)dm(x) = \frac{1}{\mu} \int_0^t h(t)dt$$

where $m(x) = \sum_{n=1}^{\infty} F_n(x)$

$$\mu = \int_0^\infty x f(x)dx$$

Note: Blackwell theorem and the key renewal theorem can be shown to be equivalent.

Proof. [Homework]
Regenerative Processes

A stochastic process $X(t), t \geq 0$ with state space $\{0, 1, 2, \ldots\}$ is called a **regenerative process** if there exists a time $S_1$ such that the continuation of the process beyond is a probabilistic replica of the whole process starting at 0.

$\Rightarrow m(t)$ is a regenerative process
Renewal Theory

- The main tool for studying regenerative processes in the absence of further properties

- To study \( m(t) = i \) (e.g. expected number of customers in the system at time \( t = i \))
  
  \[ m(t) = \int_0^\infty E[N(t)|X_1 = x]dF(x) = ? \]
  
  \[ \lim_{t \to \infty} m(t) = ? \]

  Conditioning the event \( m(t) \) on the time \( X_1 \),

\[ X_1 = x \quad \rightarrow \quad X_1 = x \]

\[ 0 \quad \rightarrow \quad \text{[case 1]} \quad \rightarrow \quad \text{[case 2]} \]

\( \therefore \) \( m(t) \) is a regenerative process,

\( \therefore \) \( \hat{m}(t) (\triangleq m(X_1 + t)) \) has the same probability law as \( m(t) \)
Renewal Theory (con’t)

- Case 1: if $X_1 = x \leq t \Rightarrow E[N(t)|X_1 = x] = 1 + m(t - x)$
- Case 2: if $X_1 = x > t \Rightarrow E[N(t)|X_1 = x] = 0$

$\Rightarrow m(t) = \int_0^t [1 + m(t - x)] \, dF(x) = F(t) + \int_0^t m(t - x) \, dF(x)$

A generalization of the renewal equation is of the form

$g(t) = h(t) + \int_0^t g(t - x) \, dF(x),$

which is called a renewal-type equation.

- **Question 1.** $g(t) =$? \hspace{1cm} ($F(x)$ and $h(t)$ are known)

- **Question 2.** $\lim_{t \to \infty} g(t) =$? \hspace{1cm} (Key Renewal Theorem !!)
Renewal Theory (con’t)

**Question 1.** \( g(t) = ? \)
How to remove the recursive relationship in the renewal-type equation? Or how to obtain the solution of the renewal-type equation?

**Solution.**
Take Laplace transform and invert it or iterate the equation.

The solution of the renewal-type equation:

\[
g(t) = h(t) + \int_0^t h(t - x) \, dm(x), \quad \text{where } m(x) = \sum_{n=1}^{\infty} F_n(x)
\]

**Proof.**

\[
g = h + g \ast F
\]

\[
\Rightarrow \tilde{g}(s) = \tilde{h}(s) + \tilde{g}(s)\tilde{F}(s)
\]

Laplace Transform on both sides
Renewal Theory (con’t)

\[ \implies \hat{g}(s) = \frac{\hat{h}(s)}{1 - \hat{F}(s)} \]

\[ \implies \hat{g}(s) = \hat{h}(s)(1 + \frac{\hat{F}(s)}{1 - \hat{F}(s)}) \]

\[ \implies \hat{g}(s) = \hat{h}(s) + \hat{h}(s)\hat{m}(s) \]

\[ \implies g(t) = h(t) + \int_0^t h(t - x) \, dm(x) \]

---

**renewal-type equation**

\[ g(t) = h(t) + \int_0^t g(t - x) \, dF(x) \]

\[ \implies g(t) = h(t) + \int_0^t h(t - x) \, dm(x) \]

---

\[ m(t) = F(t) + \int_0^t m(t - x) \, dF(x) \]

\[ \implies m(t) = F(t) + \int_0^t F(t - x) \, dm(x) \]
\[ g(t) = h(t) + \int_0^t h(t - x) \, dm(x) \]

**Question 2.** \( \lim_{t \to \infty} g(t) = ? \) \[ \lim_{t \to \infty} \int_0^t h(t - x) \, dm(x) = \frac{1}{\mu} \int_0^t h(t) \, dt \]

**Solution.**

If \( h \) is directly Riemann integrable and \( F \) nonlattice with finite mean, one can then apply the *key renewal theorem* to obtain

\[ \lim_{t \to \infty} g(t) = \frac{\int_0^\infty h(t) \, dt}{\mu} \]

What about \( \lim_{t \to \infty} m(t) \)?

\[ m(t) = F(t) + \int_0^t F(t - x) \, dm(x) \]

**Is \( F(x) \) directly Riemann integrable?**

1. \( h(t) \geq 0 \) for all \( t \geq 0 \)
2. \( h(t) \) is non-increasing
3. \( \int_0^\infty h(t) \, dt < \infty \)

**Elementary Renewal Theorem**

\[ \frac{m(t)}{t} \to \frac{1}{\mu} \quad \text{as} \quad t \to \infty \]
Example

- $X = \{X_i\}$ i.i.d. inter-arrival time, mean $\mu$,
- Recall: $E[S_{N(t)+1}] = \mu[m(t) + 1]$
- Prove it using Renewal-Type Equation and its solution.

Proof.

If $\{X_i\}$ are identically distributed random variables with a common mean $\mu$, and $N$ is independent of $\{X_i\}$, then we know that $E[S_N] = \mu E[N]$, where $S_N = X_1 + \ldots + X_N$.

In a renewal process, we see that $S_{N(t)} = X_1 + \ldots + X_{N(t)}$ denotes the time of the last renewal before $t$. Unfortunately, it is not true that $E[S_{N(t)}] = \mu E[N(t)] = \mu m(t)$

This is because $N(t)$ depends on $\{X_i\}$. 
Proof

However, the following related result holds

\[ E[S_{N(t)+1}] = \mu(m(t) + 1). \]

To establish this result, we let \( g(t) = E[S_{N(t)+1}] \). Conditioning on the epoch of the first arrival, we write

\[
E[S_{N(t)+1}|X_1 = x] = \begin{cases} 
  x & x > t \\
  x + g(t - x) & x \leq t.
\end{cases}
\]

Applying the law of total probability, we find

\[
g(t) = \int_0^\infty E[S_{N(t)+1}|X_1 = x] f(x) \, dx \\
= \int_0^\infty x f(x) \, dx + \int_0^t g(t - x) f(x) \, dx
\]
\[ g(t) = h(t) + \int_0^t g(t - x) \, d F(x), \]

\[ \Rightarrow g(t) = h(t) + \int_0^t h(t - x) \, d m(x), \quad \text{where } m(x) = \sum_{n=1}^{\infty} F_n(x) \]
Corollary. If $\mu < \infty$, then

$$E[S_{N(t)+1}] = \mu[m(t) + 1]$$

Proof.

Let's stop at the first renewal after $t$, i.e., at $N(t) + 1$ renewal. To verify $N(t) + 1$ is a stopping time for the sequence of $X_i$,

$$N(t) + 1 = n \iff N(t) = n - 1 \iff X_1 + \ldots + X_{n-1} \leq t, X_1 + \ldots + X_n > t$$

$\{N(t) + 1 = n\}$ depends only on $X_1, \ldots, X_n$ and is independent of $X_{n+1}$

$\Rightarrow N(t) + 1$ is a stopping time. From Wald's equation, when $E[X] < \infty$,

$$E[X_1 + \ldots X_{N(t)+1}] = E[X] E[N(t) + 1] \iff E[S_{N(t)+1}] = \mu[m(t) + 1]$$
A stochastic process $X(t), t \geq 0$ with state space $\{0, 1, 2, \ldots\}$ is called a **regenerative process** if there exists a time $S_1$ such that the continuation of the process beyond is a probabilistic replica of the whole process starting at 0.

**Example:**

- An alternating renewal process having ON and OFF states is a regenerative process with state 1 and 0.

- If $X(t)$ denotes the number of customers in system of $M/G/1$ queue, then is a regenerative process with state $\{0, 1, 2, \ldots\}$. The regenerative point is the time the arrival finds the system empty.
Theorem

Suppose that \( \{X(t), t \geq 0\} \) is a regenerative process. If \( S_1 \) follows \( F \), non-lattice and \( E[S_1] < \infty \) then

\[
P_j = \lim_{t \to \infty} P[X(t) = j] = \frac{E[\text{amount of time in state } j \text{ during a cycle}]}{E[\text{regenerative cycle}]} = \frac{\int_0^\infty P[X(t) = j, S_1 > t]dt}{E[S_1]}
\]